Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

Transformations of Planar Nets.

BY LUTHER PFAHLER EISENHART.

1. **Introduction.** When a surface is referred to a conjugate system of curves, u and v being the parameters, the four homogeneous point coordinates, $x^{(1)}$, $x^{(2)}$, $x^{(3)}$, $x^{(4)}$, are solutions of an equation of the form

$$\frac{\partial^2\theta}{\partial u\partial v}=a\frac{\partial\theta}{\partial u}+b\frac{\partial\theta}{\partial v}+c\theta, \tag{1}$$

where a, b, c are functions of u and v. We say that the parametric curves $u=$const., $v=$const. form a net $N(x)$, x indicating any of the four coordinates, and call (1) *the point equation* of $N(x)$.

If θ is any other solution of (1), the equations

$$\frac{\partial x_1}{\partial u}=h\frac{\partial}{\partial u}\left(\frac{x}{\theta}\right),\quad \frac{\partial x_1}{\partial v}=l\frac{\partial}{\partial v}\left(\frac{x}{\theta}\right), \tag{2}$$

satisfy the condition of integrability, provided that h and l are any pair of solutions * of

$$\frac{\partial h}{\partial v}=\left(\frac{\partial\log\theta}{\partial v}-a\right)(h-l),\quad \frac{\partial l}{\partial u}=\left(\frac{\partial\log\theta}{\partial u}-b\right)(l-h). \tag{3}$$

Moreover, the four functions x_1 obtained by quadratures from (2) when x takes on the four values of the coordinates of $N(x)$ are solutions of the same equation

$$\frac{\partial^2\theta_1}{\partial u\partial v}+\frac{l}{h}\left(\frac{\partial\log\theta}{\partial v}-a\right)\frac{\partial\theta_1}{\partial u}+\frac{h}{l}\left(\frac{\partial\log\theta}{\partial u}-b\right)\frac{\partial\theta_1}{\partial v}=0. \tag{4}$$

Hence these four quantities x_1 are the homogeneous point coordinates of a net $N_1(x_1)$.

The points F_1 and F_2, with respective homogeneous coordinates

$$\theta x_1-hx,\quad \theta x_1-lx, \tag{5}$$

lie on the line joining corresponding points, M and M_1, of the nets N and N_1, and are such that as u or v varies, F_1 or F_2, respectively, moves tangentially to the line. Hence the ruled surfaces $u=$const., $v=$const. of the congruence G of lines MM_1 are its developables. We say that the nets N and N_1 are *conjugate* to the congruence G, since the developables of the latter meet the surfaces on which N and N_1 lie in these nets. Two nets so related to a congruence are said to be in the relation of a *transformation* T with one another. Conversely, any two nets so related can be defined analytically by (2) and (3), as we have shown previously.*

---

* We exclude the case $h=l=$const., since in this case the two surfaces are homothetic with respect to the origin.

In § 2 we show that if two nets N and N_1 are in the relation of a transformation T, the same is true of their first Laplace transforms, and likewise their minus first Laplace transforms. This result holds for nets in space of any order.

Any three of the functions x are the homogeneous coordinates of the planar net, say $P(x)$, into which $N(x)$ is projected from a suitable vertex of the coordinate tetrahedron upon the opposite face. Hence the preceding results can be applied to the transformation of planar nets. The lines of the congruence G project into lines of the plane, and thus the developables of the congruence lose their significance, and these transformations in general are not of much interest. However, there are certain types of planar nets characterized, for example, by geometric properties. The above general method can be used to obtain transformations of nets of a type into nets of the same type.

We consider transformations K of planar nets whose equation (1) has equal invariants into nets of the same kind. In a previous paper † we have shown how one can determine by quadratures a surface S whose asymptotic curves project into any given planar net with equal invariants. We show in § 4 that a transformation K of the planar net is equivalent to the determination of a surface S_1 such that S and S_1 are the focal surfaces of a W-congruence. We make use of these results in § 5 to determine the equations of a W-congruence.

There are certain planar nets with equal invariants which are reproduced after three transformations of Laplace. We refer to them as *nets of period* 3. They are of three types. The remainder of the paper is devoted to a study of the transformations K of these nets into nets of the same kind. A theorem of permutability of these transformations is established. We discover also another transformation of nets of period 3, purely analytic in character, and find that their determination and that of transformations K reduce to equivalent analytical problems.

---

* *Transactions of the American Mathematical Society,* Vol. XVIII (1917), pp. 97–124.

† *Annals of Mathematics,* Series 2, Vol. XVIII (1917), pp. 221–225.

In § 12 we show that the preceding results can be interpreted as giving transformations of certain surfaces discovered by Tzitzeica.

2. **Laplace Transformations and Transformations T.** The tangents to the curves $v=$const. of a net N are tangent to the curves $u=$const. of a net $(N)_1$, and the tangents to the curves $u=$const. of N are tangent to the curves $v=$const. of a net $(N)_{-1}$. The nets $(N)_1$ and $(N)_{-1}$ are called the *first* and *minus first Laplace transforms* of N. Their respective homogeneous coordinates y and z can be given the forms

$$y=\frac{\partial x}{\partial u}-bx, \quad z=\frac{\partial x}{\partial v}-ax.^*$$

We shall prove the theorem:

If N_1 is a T transform of N determined by a function θ, the first and minus first Laplace transforms of N_1 are T transforms of the first and minus first Laplace transforms of N by means of the respective functions

$$\frac{\partial \theta}{\partial u}-b\theta, \quad \frac{\partial \theta}{\partial v}-a\theta.$$

In consequence of (4) it follows that the homogeneous coordinates of the first Laplace transform of N_1 are of the form

$$\frac{\partial x_1}{\partial u}+\frac{h}{l}\left(\frac{\partial \log \theta}{\partial u}-b\right)x_1.$$

By means of (2) this is reducible to such a form that we can take for the coordinates y_1 of this transform the expressions

$$y_1=\frac{l\dfrac{\partial}{\partial u}\left(\dfrac{x}{\theta}\right)}{\dfrac{\partial}{\partial u}\log \theta-b}+x_1.$$

Making use of the fact that x and θ are solutions of equation (1), we obtain

$$\frac{\partial y_1}{\partial u}=\bar{h}\frac{\partial}{\partial u}\left(\frac{y}{\bar{\theta}}\right), \quad \frac{\partial y_1}{\partial v}=\bar{l}\frac{\partial}{\partial v}\left(\frac{y}{\bar{\theta}}\right),$$

where

$$\bar{\theta}=\frac{\partial \theta}{\partial u}-b\theta, \quad \bar{h}=\frac{\dfrac{\partial \log \theta}{\partial u}-b}{k}+l,$$

---

[*] Darboux, *Leçons*, Vol. II, pp. 27, 28.

k being the invariant of equation (1) defined by

$$k=\frac{\partial b}{\partial v}-ab-c.$$

Since the above equations necessarily satisfy the conditions of integrability, we have proved that the two nets are in the relation of a transformation T. By similar means we show that the minus first transforms are so related.

3. **Equations of Transformations** T. The three homogeneous coordinates $x^{(1)}$, $x^{(2)}$, $x^{(3)}$ of a planar net $P(x)$ are solutions not only of equation (1), but also of two equations of the form

$$\frac{\partial^2\theta}{\partial u^2}=a'\frac{\partial\theta}{\partial u}+b'\frac{\partial\theta}{\partial v}+c'\theta,\quad \frac{\partial^2\theta}{\partial v^2}=a''\frac{\partial\theta}{\partial u}+b''\frac{\partial\theta}{\partial v}+c''\theta, \tag{6}$$

where the coefficients are functions of u and v, which must be such that the conditions of integrability

$$\frac{\partial}{\partial v}\left(\frac{\partial^2 x}{\partial u^2}\right)=\frac{\partial}{\partial u}\left(\frac{\partial^2 x}{\partial u\partial v}\right),\quad \frac{\partial}{\partial u}\left(\frac{\partial^2 x}{\partial v^2}\right)=\frac{\partial}{\partial v}\left(\frac{\partial^2 x}{\partial u\partial v}\right),$$

are satisfied by all three functions x. This leads to the following six conditions:

$$\left.\begin{aligned}
&\frac{\partial a'}{\partial v}+b'a''=\frac{\partial a}{\partial u}+ab+c, \quad \frac{\partial b'}{\partial v}+a'b+b'b''+c'=\frac{\partial b}{\partial u}+ab'+b^2,\\
&\frac{\partial c'}{\partial v}+a'c+b'c''=\frac{\partial c}{\partial u}+ac'+bc, \quad \frac{\partial b''}{\partial u}+b'a''=\frac{\partial b}{\partial v}+ab+c,\\
&\frac{\partial a''}{\partial u}+a'a''+ab''+c''=\frac{\partial a}{\partial v}+a''b+a^2, \quad \frac{\partial c''}{\partial u}+a''c'+b''c=\frac{\partial c}{\partial v}+ac+bc''.
\end{aligned}\right\} \tag{7}$$

By making use of equations (6), we are able to determine functions k, m, n so that, for all three x's,

$$x_1=kx+m\frac{\partial x}{\partial u}+n\frac{\partial x}{\partial v}. \tag{8}$$

In fact, when this expression is substituted in (2), we find that k, m, n must satisfy the completely integrable system of equations

$$\left.\begin{aligned}
&\frac{\partial k}{\partial u}+mc'+nc+\frac{h}{\theta^2}\frac{\partial\theta}{\partial u}=0, \quad \frac{\partial m}{\partial u}+k+ma'+na-\frac{h}{\theta}=0,\\
&\frac{\partial n}{\partial u}+mb'+nb=0, \quad \frac{\partial k}{\partial v}+mc+nc''+\frac{l}{\theta^2}\frac{\partial\theta}{\partial v}=0,\\
&\frac{\partial m}{\partial v}+ma+na''=0, \quad \frac{\partial n}{\partial v}+k+mb+nb''-\frac{l}{\theta}=0.
\end{aligned}\right\} \tag{9}$$

4. **Planar Nets with Equal Invariants.** When $P(x)$ and $P_1(x_1)$ are the projections of two nets $N(x)$ and $N_1(x_1)$ in the relation of a transformation T, the points of the plane whose corresponding coordinates are given by (5) are points of contact of the line $\overline{PP_1}$ with its envelope as u or v varies. We consider the case when these points are harmonic to P and P_1, and say that P and P_1 are in the relation of a *transformation* K. From (5) it follows that we must have $l=-h$, and consequently from (3) that

$$\frac{\partial a}{\partial u}=\frac{\partial b}{\partial v},$$

that is, equation (1) has equal invariants.* By making a suitable choice of the homogeneous coordinates x in this case we can take equation (1) in the form

$$\frac{\partial^2\theta}{\partial u\partial v}=c\theta. \tag{10}$$

Now, in all generality we have from (3)

$$h=-l=-\theta^2, \tag{11}$$

so that, if x_1 in (2) be replaced by $x_1\theta$, we have

$$\frac{\partial}{\partial u}(x_1\theta)=-\theta\frac{\partial x}{\partial u}+x\frac{\partial\theta}{\partial u},\quad \frac{\partial}{\partial v}(x_1\theta)=\theta\frac{\partial x}{\partial v}-x\frac{\partial\theta}{\partial v}, \tag{12}$$

and the point equation of N_1 is

$$\frac{\partial^2\theta_1}{\partial u\partial v}=\theta\frac{\partial^2}{\partial u\partial v}\frac{1}{\theta}\cdot\theta_1. \tag{13}$$

We have shown† that if we define three functions $y^{(1)}$, $y^{(2)}$, $y^{(3)}$ by

$$\frac{\partial y^{(1)}}{\partial u}=x^{(2)}\frac{\partial x^{(3)}}{\partial u}-x^{(3)}\frac{\partial x^{(2)}}{\partial u},\quad \frac{\partial y^{(1)}}{\partial v}=-x^{(2)}\frac{\partial x^{(3)}}{\partial v}+x^{(3)}\frac{\partial x^{(2)}}{\partial v}, \tag{14}$$

and by similar equations obtained by permuting the hyperscripts cyclically, the function $x^{(4)}$ given by

$$x^{(4)}=x^{(1)}y^{(1)}+x^{(2)}y^{(2)}+x^{(3)}y^{(3)}, \tag{15}$$

and $x^{(1)}$, $x^{(2)}$, $x^{(3)}$ are the homogeneous point coordinates of a surface S, upon which the parametric curves are the asymptotic lines. These lines and the planar net are in perspective relation from the point (0, 0, 0, 1).

---

* This is in accordance with the similar result for nets in space first proved by Koenig's *Comptes Rendus,* Vol. CXIII (1891), p. 1022.

† *Annals, loc. cit.* The letters x and y must be interchanged to give the equations in use in the present paper.

Since $x^{(1)}$, $x^{(2)}$, $x^{(3)}$ are solutions of (10), equations (14) are of the form of equations of Lelieuvre* $y^{(1)}$, $y^{(2)}$, $y^{(3)}$, 1, are the point coordinates of a surface Σ referred to its asymptotic lines. Moreover, from (15) it is seen that Σ and S are polar reciprocal with respect to the quadric

$$Z^{(1)^2}+Z^{(2)^2}+Z^{(3)^2}=Z^{(4)^2}. \tag{16}$$

By means of the three functions $x_1^{(1)}$, $x_1^{(2)}$ $x_1^{(3)}$ obtained from the quadratures (12), we define three functions $y_1^{(1)}$, $y_1^{(2)}$, $y_1^{(3)}$ by

$$\frac{\partial y_1^{(1)}}{\partial u}=x_1^{(2)}\frac{\partial x_1^{(3)}}{\partial u}-x_1^{(3)}\frac{\partial x_1^{(2)}}{\partial u}, \quad \frac{\partial y_1^{(1)}}{\partial v}=-x_1^{(2)}\frac{\partial x_1^{(3)}}{\partial v}+x_1^{(3)}\frac{\partial x_1^{(2)}}{\partial v}, \tag{17}$$

and similar equations obtained by permuting the hyperscripts cyclically. Now, as seen above, the functions $y_1^{(1)}$, $y_1^{(2)}$, $y_1^{(3)}$, 1, are the point coordinates of a surface Σ_1, referred to its asymptotic lines. Moreover, it is readily found† that

$$y_1^{(1)}=y^{(1)}+x^{(2)}x_1^{(3)}-x^{(3)}x_1^{(2)}. \tag{18}$$

From (14) and (15) it follows that the equation of the tangent plane to Σ is

$$x^{(1)}z^{(1)}+x^{(2)}z^{(2)}+x^{(3)}z^{(3)}-x^{(3)}=0,$$

the z's being current coordinates. It is readily seen that this plane passes through the point ($y_1^{(1)}$, $y_1^{(2)}$, $y_1^{(3)}$, 1) of Σ_1. In like manner we show that the tangent plane to Σ_1 passes through the corresponding point on Σ. Hence Σ and Σ_1 are the focal surfaces of the congruence of lines joining corresponding points of these surfaces, which accordingly is a W-congruence.

By means of $y_1^{(1)}$, $y_1^{(2)}$, $y_1^{(3)}$ we obtain, as by (15), the surface S_1 whose asymptotic curves project from (0, 0, 0, 1) into the net $P_1(x_1)$. Since S_1 is the polar transform of Σ_1 with respect to the quadric (16), and this transformation changes a W-congruence into a W-congruence, we have the theorem:

When two planar nets with equal invariants are in the relation of a transformation K, the two surfaces, whose asymptotic lines are the perspectives of the curves of the nets from a point, are the focal surfaces of a W-congruence.

5. Equations of a *W*-congruence.

It is well known that the direction-cosines of the normal to a surface referred to its asymptotic lines are solutions of the same equation of Laplace with equal invariants.‡ It is possible to choose a suitable factor so that the direction-parameters (that is, quantities proportional to the direction-cosines) shall satisfy by an equation of the form (10).

---

*E., p. 193. A reference of this sort is to the author's "Differential Geometry."

† E., p. 418.

‡ E., p. 194.

In this case we say that the parameters are in the *normal form.* From (14) it is seen that $x^{(1)}$, $x^{(2)}$, $x^{(3)}$ are the direction-parameters of Σ in the normal form.

From (14) we have, by differentiation and reduction, with the aid of (6),

$$\frac{\partial^2 y^{(1)}}{\partial u^2}=a'\frac{\partial y^{(1)}}{\partial u}-b'\frac{\partial y^{(1)}}{\partial v},\quad \frac{\partial^2 y^{(1)}}{\partial v^2}=-a''\frac{\partial y^{(1)}}{\partial u}+b''\frac{\partial y^{(1)}}{\partial v}, \tag{19}$$

and

$$\frac{\partial^2 y^{(1)}}{\partial u \partial v}=\frac{\partial \log x^{(3)}}{\partial v}\frac{\partial y^{(1)}}{\partial u}+\frac{\partial \log x^{(3)}}{\partial u}\frac{\partial y^{(1)}}{\partial v}. \tag{20}$$

The coordinates $y^{(2)}$, $y^{(3)}$ also satisfy (19), and $y^{(2)}$ satisfies (20). Hence we have the theorem:

When the asymptotic lines on a surface are projected orthogonally on a plane, taken as a coordinate plane, the equation satisfied by the cartesian coordinates of this planar net is (20), *where* $x^{(3)}$ *is the direction-parameter in the normal form of the normal to the surface with respect to the normal to the plane.*

When two surfaces Σ and Σ_1 are the focal surfaces of a W-congruence, we say that they are in the relation of a *W-transformation.* Equations (18) define the general transformation. They can be given another form by means of equations (8) for the case of transformations K. In accordance with § 3 we replace x_1 and k by $x_1\theta$ and $k\theta$, respectively, and equation (8) becomes

$$x_1=kx+\frac{m}{\theta}\frac{\partial x}{\partial u}+\frac{n}{\theta}\frac{\partial x}{\partial v}. \tag{21}$$

Since now $a=b=0$ and equations (11) hold, equations (9) become

$$\left.\begin{aligned}
&\theta\frac{\partial k}{\partial u}+(k-1)\frac{\partial \theta}{\partial u}+mc'+nc=0, \quad \theta\frac{\partial k}{\partial v}+(k+1)\frac{\partial \theta}{\partial v}+mc+nc''=0,\\
&\frac{\partial m}{\partial u}+k\theta+ma'+\theta=0, \quad \frac{\partial m}{\partial v}+na''=0,\\
&\frac{\partial n}{\partial u}+mb'=0, \quad \frac{\partial n}{\partial v}+k\theta+nb''-\theta=0.
\end{aligned}\right\} \tag{22}$$

Substituting the values from (21) in (18), we get the desired equations of transformation, namely

$$y_1=y+\frac{m}{\theta}\frac{\partial y}{\partial u}-\frac{n}{\theta}\frac{\partial y}{\partial v}. \tag{23}$$

Hence the problem of W-transformations is equivalent to the solution of equation (10) and of the completely integrable system (22).

6. Nets of Period 3. We apply the preceding methods to nets for which $a=b=0$, $c'=c''=0$. These nets are characterized by the property that any one of them is reproduced after three transformations of Laplace. In this sense they are of period 3.

It follows from (7) that in the present case

$$a'=\frac{\partial \log c}{\partial u}, \quad b''=\frac{\partial \log c}{\partial v}, \quad b'=\frac{U}{c}, \quad a''=\frac{V}{c},$$

where U and V are arbitrary functions of u and v respectively. One or both of these functions can be taken equal to zero. If they are not zero, by a suitable choice of the parameters u and v of the nets we can take U and V equal to unity. Hence in all generality we have

$$b'=\frac{\varepsilon}{c}, \quad a''=\frac{\eta}{c},$$

where ε and η take the values 0 or 1, and the differential equations of the net are

$$\frac{\partial^2 x}{\partial u^2}=\frac{\partial \log c}{\partial u}\frac{\partial x}{\partial u}+\frac{\varepsilon}{c}\frac{\partial x}{\partial v}, \quad \frac{\partial^2 x}{\partial u \partial v}=cx, \quad \frac{\partial^2 x}{\partial v^2}=\frac{\eta}{c}\frac{\partial x}{\partial u}+\frac{\partial \log c}{\partial v}\frac{\partial x}{\partial v}. \tag{24}$$

All the conditions (7) are satisfied provided that c satisfies

$$\frac{\partial^2}{\partial u \partial v}\log c=c-\frac{\varepsilon\eta}{c^2}. \tag{25}$$

For the present case equations (22) reduce to

$$\left.\begin{aligned}
&\theta\frac{\partial k}{\partial u}+nc+(k-1)\frac{\partial \theta}{\partial u}=0, \quad \theta\frac{\partial k}{\partial v}+mc+(k+1)\frac{\partial \theta}{\partial v}=0,\\
&\frac{\partial m}{\partial u}+k\theta+m\frac{\partial \log c}{\partial u}+\theta=0, \quad \frac{\partial m}{\partial v}+\frac{n\eta}{c}=0,\\
&\frac{\partial n}{\partial u}+\frac{m\varepsilon}{c}=0, \quad \frac{\partial n}{\partial v}+k\theta+n\frac{\partial \log c}{\partial v}-\theta=0.
\end{aligned}\right\} \tag{26}$$

Eliminating k from the second and third of these equations, we get

$$\frac{\partial^2 m}{\partial u \partial v}+\frac{\partial \log c}{\partial u}\frac{\partial m}{\partial v}+\left(\frac{\partial^2 \log c}{\partial u \partial v}-c\right)m=0.$$

It is readily seen that $\frac{\partial \theta}{\partial v}/c$ is a solution of this equation, so that we replace m by $\mu\frac{\partial \theta}{\partial v}/c$. Similarly, we replace n by $\nu\frac{\partial \theta}{\partial u}/c$. Then (21) becomes

$$x_1=xk+\frac{\mu}{c\theta}\frac{\partial \theta}{\partial v}\frac{\partial x}{\partial u}+\frac{\nu}{c\theta}\frac{\partial \theta}{\partial u}\frac{\partial x}{\partial v}, \tag{27}$$

and (26) reduces to

$$\left.\begin{aligned}
&\frac{\partial k}{\partial u}+(k+\nu-1)\frac{1}{\theta}\frac{\partial\theta}{\partial u}=0, \quad \frac{\partial k}{\partial v}+(k+\mu+1)\frac{1}{\theta}\frac{\partial\theta}{\partial v}=0,\\
&\frac{\partial\mu}{\partial u}\frac{\partial\theta}{\partial v}+c\theta(k+\mu+1)=0, \quad \frac{\partial\mu}{\partial v}\frac{\partial\theta}{\partial u}+c\theta(k+\nu-1)=0,\\
&\frac{\partial\nu}{\partial u}\frac{\partial\theta}{\partial u}+\nu\frac{\partial^2\theta}{\partial u^2}-\nu\frac{\partial\theta}{\partial u}\frac{\partial\log c}{\partial u}+\frac{\varepsilon\mu}{c}\frac{\partial\theta}{\partial v}=0,\\
&\frac{\partial\mu}{\partial v}\frac{\partial\theta}{\partial v}+\mu\frac{\partial^2\theta}{\partial v^2}-\mu\frac{\partial\theta}{\partial v}\frac{\partial\log c}{\partial v}+\frac{\eta\mu}{c}\frac{\partial\theta}{\partial u}=0.
\end{aligned}\right\} \tag{28}$$

If the net $N_1(x_1)$ is to be of the same kind, we must have

$$\frac{\partial^2x_1}{\partial u^2}=\frac{\partial\log c_1}{\partial u}\frac{\partial x_1}{\partial u}+\frac{U}{c_1}\frac{\partial x_1}{\partial v}, \quad \frac{\partial^2x_1}{\partial u\partial v}=c_1x_1, \quad \frac{\partial^2x_1}{\partial v^2}=\frac{V}{c_1}\frac{\partial x_1}{\partial u}+\frac{\partial\log c_1}{\partial v}\frac{\partial x_1}{\partial v}, \tag{29}$$

where, as follows from (13), c_1 is given by

$$c_1=\theta\frac{\partial^2}{\partial u\partial v}\frac{1}{\theta}=-c+\frac{2}{\theta^2}\frac{\partial\theta}{\partial u}\frac{\partial\theta}{\partial v}. \tag{30}$$

When the expression for x_1, as given by (27), is differentiated and substituted in the first and third of (29), the resulting equations are reducible by means of (24) and (28) to equations of the form

$$Ax+B\frac{\partial x}{\partial u}+C\frac{\partial x}{\partial v}=0, \quad A_1x+B_1\frac{\partial x}{\partial u}+C_1\frac{\partial x}{\partial v}=0,$$

where A, B, C, A_1, B_1, C_1 are determinate functions of u and v. As the above equations must be satisfied by $x^{(1)}$, $x^{(2)}$ and $x^{(3)}$, the coefficients must vanish identically. Hence we have the following equations of condition:

$$\left.\begin{aligned}
&U(k+\mu+1)=0, \quad V(k+\nu-1)=0,\\
&U\left(\frac{2\nu}{c\theta^2(1-k)}\frac{\partial\theta}{\partial u}\frac{\partial\theta}{\partial v}-1\right)+\varepsilon\left(1-\frac{2}{c\theta^2}\frac{\partial\theta}{\partial u}\frac{\partial\theta}{\partial v}\right)=0,\\
&V\left(\frac{2\mu}{c\theta^2(1+k)}\frac{\partial\theta}{\partial u}\frac{\partial\theta}{\partial v}+1\right)-\eta\left(1-\frac{2}{c\theta^2}\frac{\partial\theta}{\partial u}\frac{\partial\theta}{\partial v}\right)=0,\\
&\frac{\partial^2\theta}{\partial u^2}=\frac{\partial\log c}{\partial u}\frac{\partial\theta}{\partial u}+\frac{1+k}{1-k}\frac{U}{c}\frac{\partial\theta}{\partial v}, \quad \frac{\partial^2\theta}{\partial v^2}=\frac{1-k}{1+k}\frac{V}{c}\frac{\partial\theta}{\partial u}-+\frac{\partial\log c}{\partial v}\frac{\partial\theta}{\partial v}.
\end{aligned}\right\} \tag{31}$$

We break up the discussion of the problem into three cases.

7. **When $U\neq0$, $V\neq0$. Nets of the First Type.** Comparing equations (28) and (31), we find that k, μ and ν are constants such that

$$k+\mu+1=0, \quad k+\nu-1=0. \tag{32}$$

Moreover, we have

$$U=\varepsilon=1, \quad V=\eta=1.$$

If k and x_1 be replaced by $1/e$ and x_1/e, equation (27) becomes

$$x_1=x-\frac{1+e}{c\theta}\frac{\partial\theta}{\partial v}\frac{\partial x}{\partial u}-\frac{1-e}{c\theta}\frac{\partial\theta}{\partial u}\frac{\partial x}{\partial v}. \tag{33}$$

The equations of condition (28) and (31) are satisfied, provided θ is a solution of the system

$$\left.\begin{aligned}\frac{\partial^2\theta}{\partial u^2}&=\frac{\partial\log c}{\partial u}\frac{\partial\theta}{\partial u}+\frac{e+1}{e-1}\frac{\varepsilon}{c}\frac{\partial\theta}{\partial v}, \quad \frac{\partial^2\theta}{\partial u\partial v}=c\theta,\\ \frac{\partial^2\theta}{\partial v^2}&=\frac{e-1}{e+1}\frac{\eta}{c}\frac{\partial\theta}{\partial u}+\frac{\partial\log c}{\partial v}\frac{\partial\theta}{\partial v},\end{aligned}\right\} \tag{34}$$

with $\varepsilon=\eta=1$. We say that a net with equations (24) where $\varepsilon=\eta=1$ is of the *first type.*

The general solution of equations (34) involves three arbitrary constants in addition to e. Hence we have the theorem:

A planar net of period 3 of the first type admits ∞^4 transforms of the same kind.

We remark that equations (29) for N_1 are of the same form as (24).

From the results of § 2 it follows that the other two nets forming with N a closed cycle under Laplace transformations are in the relation of transformations K with the two nets forming with N_1 a closed cycle. This remark applies equally well to nets of the second and third types to be discussed in the following sections.

8. **When $U=0$, $V\neq 0$. Nets of the Second Type.** From the second of (31) it follows that $\varepsilon=0$ or $c_1=0$. Assuming $\varepsilon=0$, we find, as in the previous case, that k, μ and ν are constants in the relations (32); also that $V=\eta=1$. Now (25) becomes

$$\frac{\partial^2\log c}{\partial u\partial v}=c, \tag{35}$$

of which the general integral is

$$c=-\frac{2U'V'}{(1+UV)^2}, \tag{36}$$

where U and V are arbitrary functions of u and v respectively, and the primes indicate differentiation with respect to the argument. When $\varepsilon=0$, $\eta=1$, the net is of the *second type.*

When the value (36) of c is substituted in the second of (34), we find the general integral of the resulting equation; it is

$$\theta=\frac{U_1'}{U'}+\frac{V_1'}{V'}-2\frac{U_1V+V_1U}{1+UV'}, \tag{37}$$

where U_1 and V_1 are arbitrary functions of u and v respectively.

The fifth of (31) is $\frac{\partial^2\theta}{\partial u^2}=\frac{\partial \log c}{\partial u}\frac{\partial\theta}{\partial u}$, of which the general integral is

$$\frac{\partial\theta}{\partial u}=c\phi(v),$$

ϕ being an arbitrary function of v. When the value (37) of θ is substituted in this equation, it becomes

$$\frac{1}{U'}\left(\frac{U_1'}{U'}\right)'(1+UV)^2-2V\frac{U_1'}{U'}(1+UV)+2(U_1V^2-V_1)+2\phi(v)V'=0. \tag{38}$$

Differentiating with respect to u, we get

$$\left[\frac{1}{U'}\left(\frac{U_1'}{U'}\right)'\right]'(1+UV)^2=0.$$

Hence

$$U_1=aU^2+bU+c, \tag{39}$$

where a, b, c are constants. Substituting in (38), we find

$$a-bV+cV^2=V_1-\phi V'.$$

If we replace $\phi V'$ by V_2, so that

$$V_1=a-bV+cV^2+V_2, \tag{40}$$

equation (37) reduces to

$$\theta=\frac{V_2'}{V'}-\frac{2UV_2}{1+UV}. \tag{41}$$

The last of (31) remains to be satisfied; it may be taken in the form of the last of (34). Substituting the expression (41) for θ in this equation, we have for the determination of V_2 the equation

$$V'\left(\frac{V_2'}{V'}\right)''-V''\left(\frac{V_2'}{V'}\right)'-\frac{e-1}{e+1}V_2=0. \tag{42}$$

Thus the solution of this equation determines the transformations of nets of the second type into nets of the same kind. As before we have the theorem:

A planar net of the second type admits ∞^4 *transforms of the same kind, and their determination requires the integration of an ordinary differential equation of the third order.*

It remains for us to consider the second possibility for the satisfaction of the second of (31), namely, $\varepsilon=1$, $c_1=0$. Then, from (31) we get the two equations (32), and from the first two of (28) it follows that k, μ and ν are constants. Then the fifth equations of (28) and (31) are inconsistent.

From the results of §§ 6, 7 it follows that a net of the first or second type does not admit a transformation K into a net of the second or first type respectively.

So far as the actual determination of nets of the second type goes, it will be shown that it is a problem of the integration of an ordinary differential equation of the third order involving an arbitrary function. In fact, when in (1) and (6)

$$a=b=b'=c'=c''=0,$$

equations (7) are equivalent to

$$a'=\frac{\partial \log c}{\partial u}, \quad b''=\frac{\partial \log c}{\partial v}, \quad a''=\frac{2V_1}{c}, \quad \frac{\partial^2 \log c}{\partial u \partial v}=c,$$

where V_1 is an arbitrary function of v alone. The function c is given by (36). If U and V be taken as the parameters and be replaced by u and v, in place of (24) we have the system

$$\frac{\partial^2 x}{\partial u^2}=\frac{-2v}{1+uv}\frac{\partial x}{\partial u}, \quad \frac{\partial^2 x}{\partial u \partial v}=\frac{-2}{(1+uv)^2}x, \quad \frac{\partial^2 x}{\partial v^2}=V_1(1+uv)^2\frac{\partial x}{\partial u}-\frac{2u}{1+uv}\frac{\partial x}{\partial v}.$$

The integral of the second of these equations is of the form (37) with U and V replaced by u and v. In order that the other two equations shall be satisfied, we find, similarly to (41), that the coordinates of the net are of the form

$$x^{(i)}=\psi_i(v)-\frac{2v\psi_i}{1+uv},$$

where the functions ψ_i are linearly independent solutions of the equation

$$\psi'''-2V_1\psi=0.$$

Thus the above statement has been proved.

9. **Transformations L of Nets of the First and Second Types.** The equation (25) is such that if $c(u, v)$ is a solution, so also is $c_m \equiv c(mu, v/m)$, where m is any constant. Hence with a net N of the first or second type which has equations of the form (24) in which c is known there is associated a net N_m whose equations are

$$\frac{\partial^2 x_m}{\partial u^2}=\frac{\partial}{\partial u}\log c_m\frac{\partial x_m}{\partial u}+\frac{\epsilon}{c_m}\frac{\partial x_m}{\partial v}, \quad \frac{\partial^2 x_m}{\partial u \partial v}=c_m x_m, \quad \frac{\partial^2 x_m}{\partial v^2}=\frac{\eta}{c_m}\frac{\partial x_m}{\partial u}+\frac{\partial}{\partial v}\log c_m\frac{\partial x_m}{\partial v}.$$

As this transformation from N to N_m is suggestive of the so-called Lie transformation of pseudo-spherical surfaces,* we call it a *transformation* L_m.

---

* E., p. 289.

We return to the consideration of equations (34) and note that if we effect the change of variables given by

$$u = \sqrt[3]{\frac{e-1}{e+1}}\,u_1, \quad v = \sqrt[3]{\frac{e+1}{e-1}}\,v_1,$$

these equations reduce to

$$\frac{\partial^2\theta}{\partial u_1^2} = \frac{\partial}{\partial u_1}\log c\,\frac{\partial\theta}{\partial u_1} + \frac{\varepsilon}{c}\,\frac{\partial\theta}{\partial v_1}, \quad \frac{\partial^2\theta}{\partial u_1 \partial v_1} = c\theta, \quad \frac{\partial^2\theta}{\partial v_1^2} = \frac{\eta}{c}\,\frac{\partial\theta}{\partial u_1} + \frac{\partial}{\partial v_1}\log c\,\frac{\partial\theta}{\partial v_1}.$$

Comparing these equations with the preceding set, we see that the general solution of these equations, and, consequently, of (34), is given by

$$\theta = \sum_1^3 {}_i\, a_i x_m^{(i)}\left(\frac{u}{m},\, mv\right),$$

where a_i are constants, $x_m^{(i)}(u, v)$ are the coordinates of the net N_m and

$$m = \sqrt[3]{\frac{e-1}{e+1}}.$$

Hence we have the theorem:

The complete determination of transformations K of nets of the first and second types is the same analytical problem as the complete determination of transformations L.

10. **When $U = V = 0$. Nets of the Third Type.** We consider first the case where $c_1 \neq 0$. Then, as follows from (31), $\varepsilon = \eta = 0$. In this case the net N is said to be of the *third type.* Now the last two of (31) reduce to (34) with $\varepsilon = \eta = 0$, and c is given by (36). Hence θ is of the form (37) with U_1 given by (39) and V_1 by a similar expression, say

$$V_1 = a_1 V^2 + b_1 V + c_1.$$

Accordingly θ is reducible to the form

$$\theta = [f(1 - UV) + gU + hV]/(1 + UV),$$

where f, g, h are arbitrary constants whose form in terms of the constants $a_1, b_1, \ldots, c_1$ is unessential.

When $\varepsilon = \eta = 0$, it is not necessary to make a special choice of parameters u and v so as to reduce the equations of the net to the form (24). As the parameters are consequently undetermined, we can assume that they are chosen so that the most general form of θ is

$$\theta = [f(1 - uv) + gu + hv]/(1 + uv). \tag{43}$$

From the last two of (28) we have

$$\mu = U_1, \quad \nu = V_1,$$

where U_1 and V_1 are functions of u and v respectively. The third and fourth of (28) reduce to

$$\theta(k+\mu+1)=\frac{U_1'}{2}(h-2fu-gu^2),\quad \theta(k+\nu-1)=\frac{V_1'}{2}(g-2fv-hv^2), \tag{44}$$

expressions which are consistent with the first two of (28).

If equations (44) be subtracted from one another, we get

$$\begin{aligned}&[f(1-uv)+gu+hv](U_1-V_1+2)\\ &\qquad=\left[\frac{U_1'}{2}(h-2fu-gu^2)-\frac{V_1'}{2}(g-2fv-hv^2)\right](1+uv).\end{aligned} \tag{45}$$

Differentiating with respect to u and v, we have

$$\begin{aligned}&[(h-fu)U_1]'-[(h-gv)V_1]'-2f\\ &\qquad=\left[\frac{uU_1'}{2}(h-2fu+gu^2)\right]'-\left[\frac{vV_1'}{2}(g-2fv+hv^2)\right]'.\end{aligned}$$

This equation may be replaced by the two

$$\left.\begin{aligned}\frac{U_1'}{2}(h-2fu+gu^2)&=\left(\frac{h}{u}-f\right)U_1-2f+\alpha+\frac{\beta}{u},\\ \frac{V_1'}{2}(g-2fv+hv^2)&=\left(\frac{g}{v}-f\right)V_1+\alpha+\frac{\gamma}{v},\end{aligned}\right\} \tag{46}$$

where α, β, γ are constants. Substituting in (45), we obtain

$$\left(2f+gu-\frac{h}{u}\right)U_1-\left(2f+hv-\frac{g}{v}\right)V_1+4f+2gu+2hv-(1+uv)\left(\frac{\beta}{u}-\frac{\gamma}{v}\right)=0.$$

This equation is equivalent to

$$\begin{aligned}(h-2fu-gu^2)U_1&=(4f+\delta)u+2gu^2-\beta+\gamma u^2,\\ (g-2fv-hv^2)V_1&=\delta v-2hv^2-\gamma+\beta v^2,\end{aligned}$$

where δ is a constant. These results are consistent with (46) when $\delta=-2\alpha$, in which case we have from (44) for the determination of k,

$$k[f(1-uv)+gu+hv]=(f-\alpha)(1-uv)+(g+\gamma)u+(\beta-h)v.$$

When the constants α, β, γ satisfy the conditions

$$(\beta/h)-2=\gamma/g=-\alpha/f,$$

and only in this case, U_1 and V_1 are constants. Then k is constant, and the equations of the transformation are of the form (33). As in the other two cases, we have the theorem:

A planar net of the third type admits ∞^4 *transforms of the same kind, and they can be found without quadrature.*

We consider now the case where $c_1=0$. From (30) it follows that

$$\theta=1/(U+V),$$

where U and V are functions of u and v respectively. When this value is substituted in the second of (34), we find

$$c=2U'V'/(U+V)^2.$$

In order that this value may satisfy (25), we must have $\varepsilon\eta=0$. The last two of equations (31) reduce to

$$\frac{\partial^2\theta}{\partial u^2}=\frac{\partial\log c}{\partial u}\,\frac{\partial\theta}{\partial u},\quad \frac{\partial^2\theta}{\partial v^2}=\frac{\partial\log c}{\partial v}\,\frac{\partial\theta}{\partial v},$$

which are satisfied identically by the above values, whatever be U and V.

From the fifth and fourth of (28) we get

$$\nu=V_1,\quad k=(U+V)(V_1'/2V')-V_1+1,$$

where V_1 is an arbitrary function of v. From the second of (28) we have

$$\mu=\frac{(U+V)^2}{V'}\left(\frac{V_1'}{2V'}\right)'-\frac{V_1'}{V'}(U+V)+V_1-2.$$

These values satisfy the first and third of equations (28) identically, but in order that the last of (28) be satisfied, V_1 must be such that

$$\left(\frac{V_1'}{V'}\right)''-\frac{V''}{V'}\left(\frac{V_1'}{V'}\right)'+\eta\frac{V_1}{V'}=0.$$

The function k is constant only when V_1 is constant. Then from the above equation $\eta=0$.

The results just obtained show that nets of the second and third types admit transformation K into nets whose equations are given by (1) and (6) when

$$a=b=c=b'=c'=a''=c''=0.$$

11. **Theorem of Permutability.** In this section we establish a theorem of permutability of the transformations of nets of period 3.

We take two functions θ_i and two constants e_i $(i=1,\,2)$ satisfying the systems analogous to (34), namely

$$\left.\begin{aligned}&\frac{\partial^2\theta_i}{\partial u^2}=\frac{\partial\log c}{\partial u}\,\frac{\partial\theta_i}{\partial u}+\frac{e_i+1}{e_i-1}\,\frac{\varepsilon_i}{c}\,\frac{\partial\theta_i}{\partial v},\quad \frac{\partial^2\theta_i}{\partial u\partial v}=c\theta_i,\\ &\frac{\partial^2\theta_i}{\partial v^2}=\frac{e_i-1}{e_i+1}\,\frac{\eta_i}{c}\,\frac{\partial\theta_i}{\partial u}+\frac{\partial\log c}{\partial v}\,\frac{\partial\theta_i}{\partial v}.\end{aligned}\qquad (i=1,\,2)\right\}\quad(47)$$

Thus far we suppose ε_i and η_i capable of assuming either values 0 or 1.

By means of θ_1 and θ_2 we effect transformations of N into N_1 and N_2 respectively. From the form of (12) it follows that there exists functions θ_{12} and θ_{21} defined by

$$\frac{\partial}{\partial u}(\theta_i\theta_{ij})=-\theta_i\frac{\partial\theta_j}{\partial u}+\theta_j\frac{\partial\theta_i}{\partial u},\quad \frac{\partial}{\partial v}(\theta_i\theta_{ij})=\theta_i\frac{\partial\theta_j}{\partial v}-\theta_j\frac{\partial\theta_i}{\partial v}.\quad \left(\begin{matrix}i=1,2,\\ j=1,2,\end{matrix}\ i\neq j\right) \tag{48}$$

Evidently the constants of integration in these equations can be chosen so that

$$\theta_1\theta_{12}+\theta_2\theta_{21}=0.$$

Hereafter we assume that the functions θ_{12} and θ_{21} are paired in this way, and there are ∞^1 such pairs.

The functions θ_{12} and θ_{21} can be used to effect transformations T of N_1 and N_2 respectively into the same net N_{12}.* We seek the conditions that N_{12} shall be a net of period 3.

From the preceding investigation it follows that θ_{12} must satisfy the equations

$$\frac{\partial^2\theta_{12}}{\partial u^2}=\frac{\partial\log c_1}{\partial u}\frac{\partial\theta_{12}}{\partial u}+\frac{e_{12}+1}{e_{12}-1}\frac{\varepsilon_{12}}{c_1}\frac{\partial\theta_{12}}{\partial v},\quad \frac{\partial^2\theta_{12}}{\partial v^2}=\frac{e_{12}-1}{e_{12}+1}\frac{\eta_{12}}{c_1}\frac{\partial\theta_{12}}{\partial u}+\frac{\partial\log c_1}{\partial v}\frac{\partial\theta_{12}}{\partial v}, \tag{49}$$

where c_1 is given by (30), and ε_{12} and η_{12} are 0 or 1, as the case may be.

If equations (48) be differentiated and the expressions for $\frac{\partial^2\theta_{12}}{\partial u^2}$ and $\frac{\partial^2\theta_{12}}{\partial v^2}$ be substituted in (49), the resulting equations are reducible to

$$\left.\begin{aligned}
&\theta_{12}\left(\frac{e_1+1}{e_1-1}\varepsilon_1+\frac{e_{12}+1}{e_{12}-1}\varepsilon_{12}\right)+\frac{2\varepsilon_1}{c\theta_1}\frac{e_1+1}{e_1-1}\frac{\partial\theta_1}{\partial v}\frac{\partial\theta_2}{\partial u}-\frac{2\varepsilon_2}{c\theta_1}\frac{e_2+1}{e_2-1}\frac{\partial\theta_1}{\partial u}\frac{\partial\theta_2}{\partial v}\\
&+\theta_2\left(\frac{e_{12}+1}{e_{12}-1}\varepsilon_{12}-\frac{e_1+1}{e_1-1}\varepsilon_1\right)+\theta_1\left(\frac{e_2+1}{e_2-1}\varepsilon_2-\frac{e_{12}+1}{e_{12}-1}\varepsilon_{12}\right)\frac{\partial\theta_2}{\partial v}\Big/\frac{\partial\theta_1}{\partial v}=0,\\
&\theta_{12}\left(\frac{e_1-1}{e_1+1}\eta_1+\frac{e_{12}-1}{e_{12}+1}\eta_{12}\right)+\frac{2\eta_2}{c\theta_1}\frac{e_2-1}{e_2+1}\frac{\partial\theta_1}{\partial v}\frac{\partial\theta_2}{\partial u}-\frac{2\eta_1}{c\theta_1}\frac{e_1-1}{e_1+1}\frac{\partial\theta_1}{\partial u}\frac{\partial\theta_2}{\partial v}\\
&+\theta_2\left(\frac{e_1-1}{e_1+1}\eta_1-\frac{e_{12}-1}{e_{12}+1}\eta_{12}\right)+\theta_1\left(\frac{e_{12}-1}{e_{12}+1}\eta_{12}-\frac{e_2-1}{e_2+1}\eta_2\right)\frac{\partial\theta_2}{\partial u}\Big/\frac{\partial\theta_1}{\partial u}=0.
\end{aligned}\right\} \tag{50}$$

We consider first the case when N is of the first type. As we have seen in § 7, the nets N_1, N_2 and N_{12} are then of the first type. Thus all the ε's and η's are equal to 1. Making this substitution in (50), and eliminating θ_{12} from the resulting equations, we get

$$(e_{12}-e_2)\left(\frac{e_1+1}{e_2+1}\frac{\partial\theta_2}{\partial u}\frac{\partial\theta_1}{\partial v}-\frac{e_1-1}{e_2-1}\frac{\partial\theta_1}{\partial u}\frac{\partial\theta_2}{\partial v}\right)c_1=0.$$

---

* *Transactions, loc. cit.*, §§ 5, 11.

It is readily shown that the expression in the parenthesis vanishes only when $\theta_2/\theta_1=\text{const.}$, but in this case N_1 and N_2 are the same net. Since $c_1\neq 0$, we have $e_{12}=e_2$. Then equations (50) reduce to the single one

$$(e_1e_2-1)\theta_1\theta_{12}+\frac{(e_1+1)(e_2-1)}{c}\frac{\partial\theta_1}{\partial v}\frac{\partial\theta_2}{\partial u}-\frac{(e_2+1)(e_1-1)}{c}\frac{\partial\theta_1}{\partial u}\frac{\partial\theta_2}{\partial v}+\theta_1\theta_2(e_1-e_2)=0. \quad (51)$$

This value satisfies equations (48). Hence we have the theorem:

If N_1 and N_2 are nets of the first type, which are transforms of a net N of this type, there exists a unique net N_{12} of the first type, which is a transform of N_1 and N_2; and it can be found without quadratures.

From the equations of the theorem of permutability of transformations T * we find that for the present choice of coordinates the coordinates of N_{12} are given by

$$\theta_{12}(x_{12}-x)=\theta_2(x_2-x_1). \quad (52)$$

We suppose now that the nets N, N_1 and N_2 are of the second type. If N_{12} also is to be of this type, the ε's must have the value zero, and the η's one. Then the first of (50) is satisfied identically, and if the second be differentiated with respect to u, and use is made of (48), the result is reducible to

$$\frac{\partial\theta_1}{\partial u}\frac{e_2-e_{12}}{(e_2+1)(e_{12}+1)}=0.$$

Hence $e_{12}=e_2$ and the second of (50) reduces to (51). As this value satisfies the second of (48), we have the theorem:

If N_1 and N_2, nets of the second type, are transforms of a net N of the second type, there exists a unique net N_{12} of the second type which is their transform.

We consider finally the possibility of all four nets N, N_1, N_2, N_{12} being of the third type. In this case equations (50) are satisfied identically, and consequently all the nets N_{12} are of the third type. Let the functions θ_1 and θ_2 determining the transforms N_1 and N_2 be, according to (43),

$$\theta_i=[f_i(1-uv)+g_iu+h_iv](1+uv) \qquad (i=1,2).$$

By differentiation we have

$$\frac{\partial\theta_i}{\partial u}=\frac{-2f_iv+g_i-h_iv^2}{(1+uv)^2},\quad \frac{\partial\theta_i}{\partial v}=\frac{-2f_iu+h_i-g_iu^2}{(1+uv)^2}.$$

---

* *Transactions*, p. 107.

Substituting these values in (48) and making use of the abbreviation $(ab) \equiv a_1b_2 - a_2b_1$, we get

$$\frac{\partial}{\partial u}(\theta_1\theta_{12}) = \frac{-(fg)-(fh)v^2+(gh)v}{(1+uv)^2}, \quad \frac{\partial}{\partial v}(\theta_1\theta_{12}) = \frac{(fh)+(fg)u^2+(gh)u}{(1+uv)^2}.$$

The integral of these equations is

$$\theta_1\theta_{12} = \frac{-\frac{1}{2}(gh)(1-uv)-(fg)u+(fh)v}{1+uv} + \text{const.}$$

Hence we have the theorem:

If N is a net of the third type, and N_1 and N_2 are two transforms of this type, there can be found without quadrature ∞^1 nets N_{12} which are transforms of N_1 and N_2.

12. **Surfaces of Tzitzeica.** When the homogeneous point coordinates of a net of period 3 are chosen so that they satisfy equations (24), they are the non-homogeneous coordinates of a surface referred to its asymptotic lines. In case $\varepsilon = \eta = 0$, the surface is a central quadric. When $\eta = 1$, the surface is ruled or not according as $\varepsilon = 0$ or 1. These surfaces were discovered by Tzitzeica[*] in his search for surfaces whose total curvature at each point is in constant ratio with the fourth power of the distance of the tangent plane at the point from a point fixed in space, and constitute the complete solution of the problem.

We have seen in § 8 that the complete determination of the ruled surfaces of Tzitzeica requires the integration of an equation of the third order. Making use of this result and the expressions for the coordinates as there given, Tzitzeica showed[†] that these ruled surfaces are characterized by the property that their flexnode curve is at infinity.

Since the transformation of the planar nets as given by (33) is reciprocal in character, it follows that the equations of the inverse transformation are of the same form. Interpreted for the surface whose non-homogeneous coordinates are $x^{(1)}, x^{(2)}, x^{(3)}$, we have transformations of these surfaces into surfaces of the same kind, such that a surface and a transform are the focal surfaces of a W-congruence. Tzitzeica[‡] announced, without proof, the existence of these transformations.

PRINCETON UNIVERSITY.

---

[*] *Comptes Rendus*, Vol. CXLIV (1907), p. 1257; Vol. CXLV (1907), p. 1132. Also *Rendiconti di Palermo*, Vol. XXV (1908), pp. 180–189.

[†] *Comptes Rendus*, Vol. CXLV, *loc. cit.*

[‡] *Comptes Rendus*, Vol. CL (1910), pp. 955, 1227,